Matrix Exponential Linear Algebra X. Du

- Justification: recall the solution of $y' = ay$ is $y = Ce^{ax}$.
  - Implies that the solution of $\vec{x}' = A\vec{x}$ is $\vec{x} = e^{At}\vec{c}$.
  - Furthermore, recall that the $y(0) = C$ in the solution of $y = Ce^{ax}$ to $y' = ay$.
    - In the solution of $\vec{x} = e^{At}\vec{c}$ to $\vec{x}' = A\vec{x}$, $\vec{c} = \vec{x}(0)$
- Powers of matrices:
  - Applies only to square matrices
  - Definition:
    - $A^0 = I$ (identity matrix of the same dimension)
    - $A^1 = A$
    - $A^n = A^{n-1}A$ (recursive formula)
    - $A^n = AA...A$ (n times)
    - $A^{-n} = \left(A^{-1}\right)^n = \left(A^n\right)^{-1} = A^{-1}A^{-1}...A^{-1}$ (n times)
  - Properties:
    - $A^{m+n} = A^m A^n$.
    - $A^{mn} = \left(A^m\right)^n$
- **Matrix Exponential** (power series definition):
  - Let A be a square matrix (either real or complex)
  - $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2} + \frac{A^3}{6}...$
  - Compare to power series of $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}...$
  - $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6}...$
  - Caution: $e^{A+B} \neq e^A e^B$
- $e^{At} = \Psi \cdot \Psi(0)^{-1}$
  - Ψ is any fundamental matrix of the system $\vec{x}' = A\vec{x}$